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# Chébli–Trimèche hypergroups and $W$-type spaces ☆

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**Abstract**

The generalized Fourier transformation associated with Chébli–Trimèche hypergroups is investigated in some spaces of $W$-type introduced by Gelfand and Shilov. It is established that this transformation is an isomorphism from the space $W_{M,a}$ onto the space $W^{M^\times,1/a}$, where the function $M$ and the parameter $a$ determine the growth of the testing functions in the first space, and $M^\times$ denotes the Young dual function of $M$. The translation operator and the convolution corresponding to this transform are also studied in this class of spaces.


*Keywords:* Chébli–Trimèche hypergroups; $W$-type spaces; Generalized Fourier transformation

## 1. Introduction

A hypergroup is a locally compact space on which the space of finite regular Borel measures has a convolution structure preserving the probability measures. Such a structure can arise in several ways in harmonic analysis. Main definitions and properties concerning hypergroups can be encountered in [7,12,15].

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Chébli–Trimèche hypergroups are a class of one-dimensional hypergroups on $(0,\infty)$ associated with certain Sturm–Liouville boundary value problems. We now collect some definitions and properties about these hypergroups that will be useful in the sequel.

Suppose that $A$ is a continuous function on $[0,\infty)$ that is twice continuously differentiable on $(0,\infty)$ and that satisfies the following conditions:

(i) $A(0)=0$ and $A(x)>0$, for every $x\in(0,\infty)$,
(ii) $A$ is increasing and unbounded,
(iii) $A(x)'/A(x)=(2\mu+1)/x+B(x)$ on a neighbourhood of 0, where $\mu>-1/2$, and $B$ is an odd and $C^\infty(\mathbf{R})$ function,
(iv) $A'/A\in C^\infty(0,\infty)$ is decreasing on $(0,\infty)$ and, hence, the limit $\rho:=(1/2)\lim_{x\to\infty}(A(x)'/A(x))$ exists and $\rho\geqslant 0$.

Such a function, according to [3, Section 3.5], is called a Chébli–Trimèche function.

A hypergroup $((0,\infty),\#)$ is a Chébli–Trimèche hypergroup when there exists a Chébli–Trimèche function $A$ such that for any real-valued function $f$ on $(0,\infty)$ that is the restriction of an even nonnegative $C^\infty(\mathbf{R})$ function, the generalized translation $u(x,y)=(\tau_x f)(y)$, $x,y\in(0,\infty)$, associated with $((0,\infty),\#)$ is defined by

$$(\tau_x f)(y)=\int_0^\infty f(z)(\delta_x\#\delta_y)\,(dz),\qquad x,y\in(0,\infty),$$

and coincides with the solution of the following Cauchy problem:

$$\begin{aligned}&(\Delta_x-\Delta_y)u(x,y)=0,\\&u(x,0)=f(x),\qquad x\in(0,\infty),\\&u_y(x,0)=0,\qquad x\in(0,\infty),\end{aligned}$$

where $\Delta$ represents the singular differential operator $\Delta=-(1/A)DAD$, with $D=d/dx$. Here, for each $x\in(0,\infty)$, by $\delta_x$ we denote, as usual, the point mass measure concentred in $x$. We will write $((0,\infty),\#(A))$ to refer to the Chébli–Trimèche hypergroup associated with $A$.

Well known Chébli–Trimèche hypergroups are the following. If $A(x)=x^{2\mu+1}$, $x\in(0,\infty)$, where $\mu>-1/2$, appears the Bessel–Kingman hypergroup [16]. The Jacobi hypergroup is the Chébli–Trimèche hypergroup associated with the function $A(x)=\sinh^{2\alpha+1}x\cosh^{2\beta+1}x$, with $\alpha\geqslant\beta\geqslant-1/2$ and $\alpha\neq-1/2$ [13].

The multiplicative functions on $((0,\infty),\#(A))$ coincide with the solutions $\varphi_\lambda$, $\lambda\in\mathbf{C}$, of the initial value problem

$$\Delta_x\varphi_\lambda(x)=(\lambda^2+\rho^2)\varphi_\lambda(x),\qquad \varphi_\lambda(0)=1,\qquad \varphi_\lambda'(0)=0.$$

The generalized Fourier $\mathcal{F}_A$ associated with $((0,\infty),\#(A))$ is defined by

$$\mathcal{F}_A(f)(\lambda)=\int_0^\infty \varphi_\lambda(x)f(x)A(x)\,dx,\quad \lambda\in(0,\infty),$$

for every $f$ in the Lebesgue space $L^1((0,\infty),A(x)\,dx)$. When we consider the Bessel–Kingman or the Jacobi hypergroup the generalized Fourier transform $\mathcal{F}_A$ reduces to the Hankel transform or the Jacobi transform, respectively. A Paley–Wiener theorem for the $\mathcal{F}_A$-transformation was established in [6] and [17, Théorème 7.2]. More recently, Bloom and Xu [5] investigated the generalized Fourier transform on Schwartz type spaces.

As in [5,18], here we assume that the Chébli–Trimèche function $A$ satisfies the following additional property:

(v) there exist $\delta,R>0$ such that, for every $x\in(R,\infty)$, $A'(x)/A(x)=2\rho+e^{-\delta x}L(x)$, when $\rho>0$; or $A'(x)/A(x)=(2\alpha+1)/x+e^{-\delta x}L(x)$, if $\rho=0$, where $L$ stands for a smooth function on $(0,\infty)$ such that $(d^k/dx^k)L$ is bounded on $(0,\infty)$, for every $k\in\mathbf{N}$.

This property allows to obtain a nice behaviour near the origin and the infinity for the Harish-Chandra function that gives the inverse transformation of $\mathcal{F}_A$ (see [4,18], for instance).

Our objective in this paper is to study the generalized Fourier transformation $\mathcal{F}_A$ and the $\#(A)$-convolution on the spaces of $W$-type considered by van Eijndhoven and Kerkhof [8].

Throughout this paper to simplify we will write $\#$ and $\mathcal{F}$ instead $\#(A)$ and $\mathcal{F}_A$, respectively. By $C$ we will denote a positive constant not necessarily the same in each occurrence.

## 2. Spaces of $W$-type and the generalized Fourier transformation

In this section we study the behaviour of the generalized Fourier transformation on the spaces of $W$-type.

The spaces $W$ were introduced by Gurevich [11]. Gelfand and Shilov [10] presented the main properties of the $W$-spaces and they also analyzed the behaviour of the Euclidean Fourier transformation on those spaces. In [8], van Eijndhoven and Kerkhof investigated the Hankel transform of the even functions in $W$.

We now recall the definitions of the $W$-type spaces introduced in [8].

By $\mathcal{K}$ we will denote the set of functions constituted by all those functions $M\in C^2[0,\infty)$ such that $M(0)=M'(0)=0$, $M'(\infty)=\infty$ and $M''(x)>0$, $x\in(0,\infty)$. If $M\in\mathcal{K}$, $M^\times$ represents the Young dual function of $M$. The main properties of the functions in $\mathcal{K}$ can be encountered in [8] and [10, Chapter 1].

Assume that $M \in \mathcal{K}$ and $a > 0$.

The space $W_{M,a}$ consists of all those even functions $\phi \in C^\infty(\mathbf{R})$ such that, for every $0 < \alpha < a$ and $l \in \mathbf{N}$,

$$\sup_{x\in[0,\infty)} e^{M(\alpha x)}\left|D^l\phi(x)\right| < \infty.$$

$W_{M,a}$ is endowed with the topology generated by the family $\{p_{n,l}\}_{n,l\in\mathbf{N}}$ of seminorms, where

$$p_{n,l}(\phi) = \sup_{x\in[0,\infty)} e^{M(a\frac{n}{n+1}x)}\left|D^l\phi(x)\right|, \quad \phi \in W_{M,a} \text{ and } n, l \in \mathbf{N}.$$

Thus $W_{M,a}$ is a Fréchet space.

Bloom and Xu [5] introduced the space $\mathcal{S}_p$, $0 < p \leqslant 2$, as follows. An even function $\phi \in C^\infty(\mathbf{R})$ is in $\mathcal{S}_p$ if, and only if, for every $m, n \in \mathbf{N}$,

$$\gamma^p_{m,n}(\phi) = \sup_{x\in[0,\infty)} (1+x)^m\varphi_0(x)^{-2/p}\left|D^n\phi(x)\right| < \infty.$$

$\mathcal{S}_p$ is a Fréchet space when it is topologised by the system $\{\gamma^p_{m,n}\}_{m,n\in\mathbf{N}}$ of seminorms.

Also in [5] the authors considered the space $\mathcal{H}_p$, $0 < p \leqslant 2$, constituted of all those even functions $\Phi$ defined in the strip $G_p = \{\lambda \in \mathbf{C}\colon |\operatorname{Im}\lambda| \leqslant \rho(2/p - 1)\}$ that satisfy:

(i) $\Phi$ is holomorphic in $G_p^0 = \{\lambda \in \mathbf{C}\colon |\operatorname{Im}\lambda| < \rho(2/p - 1)\}$ and, for every $k \in \mathbf{N}$, the function $D^k\Phi(\lambda)$ can be continuously extended to $G_p$.

(ii) For every $m, n \in \mathbf{N}$,

$$\eta^p_{m,n}(\Phi) = \sup_{\lambda\in G_p} \bigl(1+|\lambda|\bigr)^m\left|D^n\Phi(\lambda)\right| < \infty.$$

$\mathcal{H}_p$ is equipped with the topology generated by $\{\eta^p_{m,n}\}_{m,n\in\mathbf{N}}$. Thus, $\mathcal{H}_p$ is a Fréchet space and the generalized Fourier transformation $\mathcal{F}$ is an isomorphism from $\mathcal{S}_p$ onto $\mathcal{H}_p$ [5, Theorem 4.27].

The space $\mathcal{S}_p$ contains continuously $W_{M,a}$. Indeed, let $m \in \mathbf{N}$ and $\alpha \in (0, a)$. According to [5, Lemma 3.4, (iii)] and [8, Lemma 2.4], we have

$$\begin{aligned}
&e^{-M(\alpha x)}(1+x)^m\varphi_0(x)^{-2/p}\\
&\quad\leqslant e^{-M(\alpha x)+2\rho x/p}(1+x)^m\\
&\quad\leqslant e^{M^\times(k/\alpha)-(k-2\rho/p)x}(1+x)^m \leqslant C, \quad x \in [0,\infty).
\end{aligned}$$

Here $k \in \mathbf{N}$ is chosen such that $k > 2\rho/p$.

Hence, by [5, Theorem 4.27], for every $\phi \in W_{M,a}$, we have that $\mathcal{F}\phi \in \mathcal{H}_p$ and

$$\phi(x) = \int_0^\infty \varphi_\lambda(x)(\mathcal{F}\phi)(\lambda)\,\frac{d\lambda}{|c(\lambda)|^2}, \quad x \in \mathbf{R}.$$

The space $W^{M,a}$ is constituted by all those even and entire functions $\Phi$ such that, for every $\beta > a$ and $l \in \mathbf{N}$,

$$\sup_{z\in\mathbf{C}} e^{-M(\beta|\operatorname{Im} z|)}\left|z^{2l}\Phi(z)\right| < \infty.$$

$W^{M,a}$ becomes a Fréchet space when we consider on $W^{M,a}$ the topology associated with the family $\{q_{n,l}\}_{n,l\in\mathbf{N}}$ of seminorms, where, for every $n, l \in \mathbf{N}$,

$$q_{n,l}(\Phi) = \sup_{z\in\mathbf{C}} e^{-M(a\frac{n+2}{n+1}|\operatorname{Im} z|)}\left|z^{2l}\Phi(z)\right|, \quad \Phi \in W^{M,a}.$$

The space $W^{M,a}$ is continuously contained in $\mathcal{H}_p$, $0 < p \leqslant 2$. Indeed, let $0 < p \leqslant 2$ and $\Phi \in W^{M,a}$. Since $\Phi$ is entire, we can write, for every $n \in \mathbf{N}$,

$$\frac{d^n}{d\lambda^n}\Phi(\lambda) = \frac{n!}{2\pi i}\int_{\mathcal{C}_\lambda} \frac{\Phi(w)}{(w-\lambda)^{n+1}}\,dw, \quad \lambda \in \mathbf{C},$$

where $\mathcal{C}_\lambda$ represents the circle centred in $\lambda$ given by the parametrization $w = \lambda + e^{it}$, $t \in [0, 2\pi)$, for every $\lambda \in \mathbf{C}$. Hence, for each $m, n \in \mathbf{N}$, we have

$$\sup_{\lambda\in G_p}\big(1+|\lambda|\big)^m\left|\frac{d^n}{d\lambda^n}\Phi(\lambda)\right| \leqslant C\sup_{\lambda\in\mathbf{C}} e^{-M(2a|\operatorname{Im}\lambda|)}\big(1+|\lambda|\big)^m\left|\Phi(\lambda)\right|.$$

Then $\Phi \in \mathcal{H}_p$. Also the last inequality proves that $W^{M,a}$ is continuously contained in $\mathcal{H}_p$.

By [5, Theorem 4.27], for every $\Phi \in W^{M,a}$, the function $\phi$ defined by

$$\phi(x) = \int_0^\infty \varphi_\lambda(x)\Phi(\lambda)\,\frac{d\lambda}{|c(\lambda)|^2}, \quad x \in \mathbf{R},$$

is in $\mathcal{S}_p$, $0 < p \leqslant 2$, and $\mathcal{F}\phi = \Phi$.

The $S$-type spaces introduced by Gelfand and Shilov [9] reduce to $W$ spaces in some cases.

If $\alpha > 0$ and $A > 0$, the space $S_{\alpha,A}$ is constituted by all those even functions $\phi \in C^\infty(\mathbf{R})$ verifying

$$\sup_{x\in\mathbf{R},\ k\in\mathbf{N}} \frac{|x^k D^m\phi(x)|}{(A+\delta)^k k^{k\alpha}} < \infty,$$

for every $\delta > 0$ and $m \in \mathbf{N}$.

$\mathcal{S}_{\alpha,A}$ coincides with the space $W_{M,a}$, when $a = 1/(Ae^\alpha)$ and $M(x) = \alpha x^{1/\alpha}$, $x \in (0, \infty)$, provided that $0 < \alpha < 1$ [9, p. 172].

For each $\beta > 0$ and $B > 0$, $S^{\beta,B}$ consists of all those even functions $\phi \in C^\infty(\mathbf{R})$ such that, for every $\xi > 0$ and $k \in \mathbf{N}$,

$$\sup_{x\in\mathbf{R},\ m\in\mathbf{N}} \frac{|x^k D^m\phi(x)|}{(B+\xi)^m m^{m\beta}} < \infty.$$

When $M(x) = (1-\beta)x^{1/(1-\beta)}$, $x > 0$, and $0 < \beta < 1$, the space $W^{M,a}$ reduces to $S^{\beta,ae^{-\beta}}$ [9, p. 210].

We now analyze the generalized Fourier transformation $\mathcal{F}$ on the spaces $W$. Previously we need to present a property of the function $\varphi_\lambda$, $\lambda \in \mathbf{C}$.

**Lemma 2.1.** *For every $n \in \mathbf{N}$ there exists $C > 0$ such that*

$$\left|\frac{\partial^n}{\partial\lambda^n}\varphi_\lambda(x)\right| \leqslant C(1+x)^{n+1}e^{(|\operatorname{Im}\lambda|-\rho)x}, \quad x \in [0,\infty) \text{ and } \lambda \in \mathbf{C}.$$

**Proof.** This property can be proved as [5, Lemma 3.4, (iv)] although there the inequality was established for $|\operatorname{Im}\lambda| < \rho$. $\square$

**Theorem 2.2.** *Let $M \in \mathcal{K}$ and $a > 0$. Then the generalized Fourier transformation defines a continuous mapping from $W_{M,a}$ into $W^{M^\times,1/a}$.*

**Proof.** Let $\phi \in W_{M,a}$ and define $\Phi = \mathcal{F}\phi$. To see that $\Phi$ is a holomorphic function on the complex plane $\mathbf{C}$, it is sufficient to note that, by Lemma 2.1 and [5, (3.5)], there exists $\beta > 0$ for which

$$\begin{aligned}
&\int_0^\infty \left|\frac{\partial}{\partial\lambda}\varphi_\lambda(x)\right|\left|\phi(x)\right|A(x)\,dx \\
&\quad\leqslant C\int_0^\infty (1+x)^\beta e^{(|\operatorname{Im}\lambda|+\rho)x}\left|\phi(x)\right|dx \leqslant C\int_0^\infty (1+x)^\beta e^{M(ax/4)}\left|\phi(x)\right|dx \\
&\quad\leqslant C\int_0^\infty (1+x)^\beta e^{-M(ax/2)}\,dx \sup_{x\in[0,\infty)} e^{M(ax/2)}\left|\phi(x)\right|.
\end{aligned}$$

Note that the last integral is finite.

Let $n \in \mathbf{N}$. Since $\phi \in \mathcal{S}_p$, for every $0 < p \leqslant 2$, one has

$$\lambda^{2n}\Phi(\lambda) = \sum_{j=0}^n \binom{n}{j}(-\rho^2)^{n-j}\int_0^\infty \varphi_\lambda(x)\Delta^j\phi(x)A(x)\,dx, \quad \lambda \in \mathbf{C}. \tag{2.1}$$

Let $j \in \mathbf{N}$. By invoking [5, Lemma 4.18, (ii)], there exist $\delta > 0$ and $s_j \in \mathbf{N}$ such that for every $x \in (0,\delta)$ we can find $\chi_l \in (0,x)$, $l = 1,2,\ldots,s_j$, for which

$$\left|\Delta^j\phi(x)\right| \leqslant C\sum_{i=1}^{2j}\left(\sum_{l=0}^{s_j}\left|D^i\phi(\chi_l)\right| + \left|D^i\phi(x)\right|\right).$$

Hence, according to Lemma 2.1, it follows, for every $\alpha > 1$,

$$\begin{aligned}
&\int_0^\delta \left|\varphi_\lambda(x)\Delta^j\phi(x)\right| A(x)\,dx \\
&\quad\leqslant C\int_0^\delta e^{(|\operatorname{Im}\lambda|-\rho)x}(1+x)A(x)\sum_{i=1}^{2j}\left(\sum_{l=0}^{s_j}\left|D^i\phi(\chi_l)\right|+\left|D^i\phi(x)\right|\right)dx \\
&\quad\leqslant Ce^{M^\times(\frac{\alpha}{a}|\operatorname{Im}\lambda|)}\sum_{i=1}^{2j}\sup_{x\in[0,\infty)} e^{M(\frac{a}{\alpha}x)}\left|D^i\phi(x)\right|,\quad \lambda\in\mathbf{C}.
\end{aligned}\tag{2.2}$$

Here $C$ is not depending on $\phi$.

Moreover, by invoking [5, Lemma 4.18, (iii), and (3.5)] we are led to

$$\begin{aligned}
&\int_\delta^\infty \left|\varphi_\lambda(x)\Delta^j\phi(x)\right| A(x)\,dx \\
&\quad\leqslant C\sum_{i=1}^{2j}\int_\delta^\infty e^{(|\operatorname{Im}\lambda|+\rho)x}(1+x)^\beta\left|D^i\phi(x)\right|dx \\
&\quad\leqslant Ce^{M^\times(\frac{\alpha}{a}|\operatorname{Im}\lambda|)}\sum_{i=1}^{2j}\int_0^\infty e^{M(\frac{a}{\alpha}x)}e^{M(\frac{\alpha-1}{4\alpha}ax)}(1+x)^\beta\left|D^i\phi(x)\right|dx \\
&\quad\leqslant Ce^{M^\times(\frac{\alpha}{a}|\operatorname{Im}\lambda|)}\sum_{i=1}^{2j}\sup_{x\in[0,\infty)} e^{M(\frac{1+\alpha}{2\alpha}ax)}\left|D^i\phi(x)\right|,\quad \lambda\in\mathbf{C},
\end{aligned}\tag{2.3}$$

for each $\alpha>1$ and some $\beta>0$.

By combining (2.1)–(2.3) we conclude that if $\alpha>1$ and $n\in\mathbf{N}$, there exists $C>0$ for which

$$\sup_{\lambda\in\mathbf{C}} e^{-M^\times(\frac{\alpha}{a}|\operatorname{Im}\lambda|)}\left|\lambda^{2n}\Phi(\lambda)\right|\leqslant C\sum_{i=1}^{2n}\sup_{x\in[0,\infty)} e^{M(\frac{1+\alpha}{2\alpha}ax)}\left|D^i\phi(x)\right|.$$

Thus we have proved that the generalized Fourier transformation is continuous from $W_{M,a}$ into $W^{M^\times,1/a}$. □

**Theorem 2.3.** *Let $M\in\mathcal{K}$ and $a>0$. Then the mapping $\mathcal{G}$ defined by*

$$\Phi\to\mathcal{G}(\Phi)(x)=\int_0^\infty \varphi_\lambda(x)\Phi(\lambda)\,\frac{d\lambda}{|c(\lambda)|^2},\quad x\in\mathbf{R},$$

*is continuous from $W^{M,a}$ into $W_{M^\times,1/a}$.*

**Proof.** To prove this proposition we will adapt an argument of Anker [1]. Let $\Phi \in W^{M,a}$ and define $\phi = \mathcal{G}\Phi$. Also we consider $\psi = \mathcal{F}_0^{-1}\Phi$, where $\mathcal{F}_0$ represents the Euclidean Fourier transformation on $\mathbf{R}$, that is, $\psi(x) = (1/2\pi)\int_{-\infty}^{\infty} e^{ixy}\Phi(y)\,dy$, $x \in \mathbf{R}$.

Let $l \in \mathbf{N}$ and $\alpha \in (0,1)$. According to [5, Lemma 3.6, (ii)], we can write

$$\left|D^l\phi(x)\right| \leqslant C(1+x)^3 e^{-\rho x}\int_0^\infty \left|\lambda^2+\rho^2\right|^m \left|\Phi(\lambda)\right| \frac{d\lambda}{|c(\lambda)|^2}, \quad x \in (0,\infty),$$

for a certain $m \in \mathbf{N}$. Moreover, by [18, p. 99], it deduces that

$$\begin{aligned}\left|D^l\phi(x)\right| &\leqslant C(1+x)^3 e^{-\rho x}\int_0^\infty \left|\lambda^2+\rho^2\right|^m \left|\Phi(\lambda)\right| \lambda^{2\mu+1}\,d\lambda \\ &\leqslant C(1+x)^3 e^{-\rho x}\Big(\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi(\lambda)\right| \\ &\qquad + \sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\Phi(\lambda)\right|\Big),\end{aligned}$$

where $s \in \mathbf{N}$, $s > m+\mu+1$ and $\beta > 1$. The number $\beta$ will be specified later.

Hence, it follows that, for every $j \in \mathbf{N}$, there exists $C_j > 0$ for which

$$\begin{aligned}&\sup_{x\in(0,j+1]} e^{M^\times(\frac{\alpha}{a}x)}\left|D^l\phi(x)\right| \\ &\quad \leqslant C_j\Big(\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi(\lambda)\right| + \sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\Phi(\lambda)\right|\Big).\end{aligned} \tag{2.4}$$

We now choose a sequence $\{w_j\}_{j\in\mathbf{N}}$ of functions such that

(i) $w_j \in C^\infty(\mathbf{R})$ is even, for every $j \in \mathbf{N}$,
(ii) $w_j(x) = 1$, $|x| \leqslant j$, and $w_j(x) = 0$, $|x| \geqslant j+1$, for each $j \in \mathbf{N}$,
(iii) for every $k \in \mathbf{N}$, there exists $C_k > 0$ for which $|D^k w_j(x)| \leqslant C_k$, $x \in \mathbf{R}$ and $j \in \mathbf{N}$.

We consider the following decomposition of $\psi$,

$$\psi = w_j\psi + (1-w_j)\psi,$$

for every $j \in \mathbf{N}$. We define, for each $j \in \mathbf{N}$, the functions $\psi_j$, $\phi_j$ and $\Phi_j$ as

$$\psi_j = (1-w_j)\psi, \qquad \Phi_j = \mathcal{F}_0\psi_j \quad \text{and} \quad \phi_j = \mathcal{G}\Phi_j.$$

Note that $w_j\psi = 0$, outside $[-j-1, j+1]$, $j \in \mathbf{N}$. Hence, according to [5, Lemma 4.11], since $\mathcal{F} = \mathcal{F}_0 \cdot \mathcal{A}$, where $\mathcal{A}$ denotes the Abel transformation defined in [5, (4.9)], we infer that $\phi = \phi_j$, outside $[-j-1, j+1]$, $j \in \mathbf{N}$.

Let $j \in \mathbf{N}\backslash\{0\}$. By proceeding as above we can obtain

$$\begin{aligned}
&\sup_{j+1\leqslant x\leqslant j+2} e^{M^\times(\frac{\alpha}{a}x)}\left|D^l\phi(x)\right| \\
&\quad\leqslant C(j+2)^3 e^{M^\times(\frac{\alpha}{a}(j+2))-\rho j}\Big(\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi_j(\lambda)\right| \\
&\qquad+\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\Phi(\lambda)\right|\Big). \qquad (2.5)
\end{aligned}$$

By invoking well known operational rules of the Euclidean Fourier transformation $\mathcal{F}_0$ we have

$$\lambda^{2s}\Phi_j(\lambda)=\int_{-\infty}^{\infty} e^{-it\lambda}D^{2s}\psi_j(t)\,dt, \quad \lambda\in\mathbf{C}.$$

Assume that $\delta>1$ such that $\alpha\delta<1$. According to [9, p. 20, Theorem 1], we get

$$\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi_j(\lambda)\right|\leqslant C\sup_{t\in(0,\infty)} e^{M^\times(\frac{\gamma}{a}t)}\left|D^{2s}\psi_j(t)\right|,$$

where $0<\gamma<1$ is depending on $\beta$. By studying the proof of [9, p. 10, Theorem 1] we can see that $\beta$ can be chosen such that $\alpha\delta+\gamma<1$.

Then the properties of $\{w_j\}_{j\in\mathbf{N}}$ lead to

$$\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi_j(\lambda)\right|\leqslant C\sum_{l=0}^{2s}\sup_{t\geqslant j} e^{M^\times(\frac{\gamma}{a}t)}\left|D^l\psi(t)\right|.$$

Hence, since the function $M^\times$ is increasing,

$$\begin{aligned}
&(j+2)^3 e^{M^\times(\frac{\alpha}{a}(j+2))-\rho j}\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi_j(\lambda)\right| \\
&\quad\leqslant (j+2)^3 e^{M^\times(\frac{\alpha\delta}{a}j)-\rho j}\sup_{\lambda\in\mathbf{C}} e^{-M(\beta a|\operatorname{Im}\lambda|)}\left|\lambda^{2s}\Phi_j(\lambda)\right| \\
&\quad\leqslant C\sum_{l=0}^{2s}\sup_{t\geqslant j} t^3 e^{M^\times(\frac{\alpha\delta}{a}t)+M^\times(\frac{\gamma}{a}t)-\rho t}\left|D^l\psi(t)\right| \\
&\quad\leqslant C\sum_{l=0}^{2s}\sup_{t\in(0,\infty)} e^{M^\times(\frac{\alpha\delta+\gamma}{a}t)}\left|D^l\psi(t)\right|, \qquad (2.6)
\end{aligned}$$

provided that $j\in\mathbf{N}$ is large enough.

Now, by [9, p. 11, Theorem 2] we conclude that

$$\sup_{t\in(0,\infty)} e^{M^\times(\frac{\alpha\delta+\gamma}{a}t)}\left|D^l\psi(t)\right|\leqslant C\sum_{k=0}^{p}\sup_{\lambda\in\mathbf{C}} e^{-M(\varepsilon a|\operatorname{Im}\lambda|)}\left|\lambda^k\Phi(\lambda)\right|, \qquad (2.7)$$

for every $l=0,1,\ldots,2s$ and a certain $\varepsilon>1$ and $p\in\mathbf{N}$.

From (2.4)–(2.7) it is inferred the desired continuity of $\mathcal{G}$. $\square$

As a consequence of Theorems 2.2 and 2.3, in line with [5, Theorem 4.27], we can establish the following result.

**Corollary 2.4.** *Let $M \in \mathcal{K}$ and $a > 0$. Then the generalized Fourier transformation $\mathcal{F}$ is an isomorphism from $W_{M,a}$ onto $W^{M^\times,1/a}$.*

**Corollary 2.5.** *If $0 < \alpha < 1$ and $A > 0$, the generalized Fourier transformation is an isomorphism from $S_{\alpha,A}$ onto $S^{\alpha,A}$.*

**Proof.** It is an immediate consequence of Corollary 2.4. □

In [2] Betancor and Rodríguez-Mesa, inspired by the studies of Pathak and Upadhyay [14], obtained new characterizations of the spaces of $W$-type in terms of the Bessel operator. Now we establish a characterization of the space $W_{M,a}$ that involves the operator $\Delta$.

Let $1 \leqslant q \leqslant \infty$. We will say that an even function $\phi \in C^\infty(\mathbf{R})$ is in $W_{M,a}^{q,\Delta}$ if, and only if, $e^{\rho x}(1+x)^{1+\beta} D\Delta^m\phi(x) \to 0$, as $x \to \infty$, for every $m \in \mathbf{N}$, where $\beta$ is given as in [5, (3.5)], that is such that $A(x) \leqslant A(1)x^\beta e^{2\rho x}$, when $x$ is large, and

$$\alpha_{m,n}^{q,\Delta}(\phi) = \left\| e^{M(a\frac{n}{n+1}x)} \Delta^m \phi(x) \right\|_q < \infty,$$

for every $m, n \in \mathbf{N}$, where $\|.\|_q$ represents the usual norm in the Lebesgue space $L_q(0,\infty)$. On $W_{M,a}^{q,\Delta}$ we consider the topology generated by the family $\{\alpha_{m,n}^{q,\Delta}\}_{m,n\in\mathbf{N}}$ of seminorms.

**Proposition 2.6.** *Let $M \in \mathcal{K}$ and $a > 0$. For every $1 \leqslant q \leqslant \infty$, $W_{M,a}^{q,\Delta} = W_{M,a}$, where the equality is algebraic and topological.*

**Proof.** Let $1 \leqslant q \leqslant \infty$. Assume that $\phi \in W_{M,a}$ and $m, n \in \mathbf{N}$. According to [5, Lemma 4.18] we can write, for certain $\delta, C > 0$ that do not depend on $\phi$,

$$e^{M(a\frac{n}{n+1}x)} \left|\Delta^m \phi(x)\right| \leqslant C \sum_{j=1}^{2m} e^{M(a\frac{n}{n+1}x)} \left|D^j \phi(x)\right|, \quad x \in (\delta, \infty),$$

and

$$e^{M(a\frac{n}{n+1}x)} \left|\Delta^m \phi(x)\right| \leqslant C \sum_{i=1}^{2m} \left( \sum_{j=1}^{s_m} e^{M(a\frac{n}{n+1}\chi_j)} \left|D^i \phi(\chi_j)\right| \right.$$
$$\left. + e^{M(a\frac{n}{n+1}x)} \left|D^i \phi(x)\right| \right), \quad x \in (0, \delta),$$

for certain $s_m \in \mathbf{N}$ and $\chi_j = \chi_j(x,m) \in [0,x]$, $j = 1, 2, \ldots, s_m$.

Hence we deduce that

$$\alpha_{m,n}^{q,\Delta}(\phi) \leqslant C \sum_{j=1}^{2m} p_{n+1,j}(\phi).$$

Moreover, by proceeding as in the proof of [5, Lemma 4.18, (iii)] we can see that

$$\left|D\Delta^m \phi(x)\right| \leqslant C \sum_{j=1}^{2m+1} \left|D^j \phi(x)\right|, \quad x \in (\delta, \infty). \tag{2.8}$$

Then by [8, Lemma 2.4] and (2.8), we obtain

$$\lim_{x\to\infty} e^{\rho x}(1+x)^{1+\beta} D\Delta^m \phi(x) = 0.$$

Thus, we prove that $W_{M,a}$ is continuously contained in $W_{M,a}^{q,\Delta}$.

Assume now that $\phi \in W_{M,a}^{q,\Delta}$ and define $\Phi = \mathcal{F}\phi$. Partial integration leads to

$$\begin{aligned}\lambda^2\Phi(\lambda) &= \int_0^\infty \Delta_x\big(\varphi_\lambda(x)\big)\phi(x)A(x)\,dx - \rho^2 \int_0^\infty \varphi_\lambda(x)\phi(x)A(x)\,dx \\ &= \int_0^\infty \varphi_\lambda(x)\Delta\big(\phi(x)\big)A(x)\,dx - \rho^2 \int_0^\infty \varphi_\lambda(x)\phi(x)A(x)\,dx, \\ &\quad \lambda \in (0,\infty).\end{aligned}$$

We have taken into account that, since $\phi \in W_{M,a}^{q,\Delta}$,

$$\lim_{x\to 0} D\big(\varphi_\lambda(x)\big)\phi(x)A(x) = \lim_{x\to 0} \varphi_\lambda(x) D\big(\phi(x)\big)A(x) = 0, \quad \lambda \in (0,\infty),$$

and

$$\lim_{x\to\infty} D\big(\varphi_\lambda(x)\big)\phi(x)A(x) = \lim_{x\to\infty} \varphi_\lambda(x)\big(D\phi(x)\big)A(x) = 0, \quad \lambda \in (0,\infty).$$

By iterating this argument we can obtain that, for every $n \in \mathbf{N}$,

$$\lambda^{2n}\Phi(\lambda) = \sum_{j=0}^{n} \binom{n}{j} (-\rho^2)^{n-j} \int_0^\infty \varphi_\lambda(x)\Delta^j\big(\phi(x)\big)A(x)\,dx, \quad \lambda \in (0,\infty). \tag{2.9}$$

Moreover, by proceeding as in the proof of Theorem 2.2 we can see that each the two sides of (2.9) defines a holomorphic function. Hence, the equality in (2.9) holds for every $\lambda \in \mathbf{C}$.

From (2.9), if $n \in \mathbf{N}$ and $\alpha > 1$, for a certain $C > 0$ we conclude that

$$\sup_{\lambda\in\mathbf{C}} e^{-M^\times(\frac{\alpha}{a}|\operatorname{Im}\lambda|)}\left|\lambda^{2n}\Phi(\lambda)\right| \leqslant C \sum_{j=1}^{n} \left\|e^{M(\frac{1+\alpha}{2\alpha}ax)}\Delta^j\phi(x)\right\|_q.$$

Then, it is established that the generalized Fourier transformation $\mathcal{F}$ maps continuously $W_{M,a}^{q,\Delta}$ into $W^{M^\times,1/a}$.

Finally, by Theorem 2.3 and [5, Theorem 4.27] we deduce that $W_{M,a}^{q,\Delta}$ is continuously contained in $W_{M,a}$.

The proof is now complete. □

Note that the last proposition is an extension of the result achieved in [2, Theorem 2.1].

## 3. The #-convolution on $W_{M,a}$ and its dual

In this section we study the #-convolution on $W_{M,a}$ and its dual space $W'_{M,a}$.

We start analyzing the behaviour of the translation operator $\tau_x$, $x \in (0,\infty)$, on $W_{M,a}$.

**Proposition 3.1.** *Let $M \in \mathcal{K}$ and $a > 0$. The translation operator $\tau_x$ defines a continuous linear mapping from $W_{M,a}$ into itself, for every $x \in (0,\infty)$.*

**Proof.** Let $x \in (0,\infty)$ and $\phi \in W_{M,a}$. Since, by [5, Theorem 2.4, (i)], we can set

$$(\tau_x\phi)(y) = \mathcal{F}^{-1}\big(\varphi_\lambda(x)\mathcal{F}(\phi)(\lambda)\big)(y), \quad y \in (0,\infty),$$

according to Corollary 2.4, the proof will be finished when we prove that the mapping defined by

$$\Phi \to \varphi_\lambda(x)\Phi(\lambda)$$

is continuous from $W^{M^\times,1/a}$ into itself.

Let $\Phi \in W^{M^\times,1/a}$. Since $\varphi_\lambda(x)$ is an even and entire function, $\varphi_\lambda(x)\Phi(\lambda)$ is also even and entire. Moreover, by [5, Lemma 3.4], for every $m,n \in \mathbf{N}$, we have

$$\begin{aligned}
&e^{-M^\times(\frac{1}{a}\frac{n+2}{n+1}|\operatorname{Im}\lambda|)}\left|\lambda^{2m}\varphi_\lambda(x)\Phi(\lambda)\right| \\
&\quad\leqslant C(1+x)e^{(|\operatorname{Im}\lambda|-\rho)x-M^\times(\frac{1}{a}\frac{n+2}{n+1}|\operatorname{Im}\lambda|)}\left|\lambda^{2m}\Phi(\lambda)\right| \\
&\quad\leqslant C(1+x)e^{-\rho x}e^{|\operatorname{Im}\lambda|x-M^\times(\frac{1}{a}\frac{n+3}{n+2}|\operatorname{Im}\lambda|)+M^\times(\frac{1}{a}\frac{n+3}{n+2}|\operatorname{Im}\lambda|)-M^\times(\frac{1}{a}\frac{n+2}{n+1}|\operatorname{Im}\lambda|)} \\
&\qquad\times\left|\lambda^{2m}\Phi(\lambda)\right| \\
&\quad\leqslant C(1+x)e^{-\rho x}e^{|\operatorname{Im}\lambda|x-M^\times(\frac{1}{a}\frac{|\operatorname{Im}\lambda|}{n^2+3n+2})}e^{-M^\times(\frac{1}{a}\frac{n+3}{n+2}|\operatorname{Im}\lambda|)}\left|\lambda^{2m}\Phi(\lambda)\right| \\
&\quad\leqslant C(1+x)e^{-\rho x+M(xa(n^2+3n+2))}q_{n+1,m}(\Phi), \quad \lambda \in \mathbf{C}.
\end{aligned}$$

Hence, we get

$$q_{n,m}\big(\varphi_\lambda(x)\Phi(\lambda)\big) \leqslant C(1+x)e^{-\rho x+M(xa(n^2+3n+2))}q_{n+1,m}(\Phi), \quad m,n \in \mathbf{N}.$$

Thus the proof is complete. □

We now study the #-convolution on the spaces $W_{M,a}$.

**Proposition 3.2.** *Let $M \in \mathcal{K}$ and $a, b > 0$. The mapping defined by $(\phi, \psi) \to \phi\#\psi$ is bilinear and continuous from $W_{M,a} \times W_{M,b}$ into $W_{M,c}$, provided that $c = ab/(a+b)$.*

**Proof.** Let $\phi \in W_{M,a}$ and $\psi \in W_{M,b}$. By resorting to [5, Theorem 2.4, (ii)], one has

$$\mathcal{F}(\phi\#\psi) = \mathcal{F}(\phi)\mathcal{F}(\psi).$$

Hence, by Corollary 2.4, the proof will finish when we prove that the mapping defined by $(\Phi, \Psi) \to \Phi\Psi$ is continuous from $W^{M,a} \times W^{M,b} \to W^{M,a+b}$. Let $m, n \in \mathbf{N}$. We can write, for every $\Phi \in W^{M,a}$ and $\Psi \in W^{M,b}$,

$$\begin{aligned} q_{m,n}(\Phi\Psi) &= \sup_{\lambda\in\mathbf{C}} e^{-M((a+b)\frac{m+2}{m+1}|\operatorname{Im}\lambda|)} \left|\lambda^{2m}\Phi(\lambda)\Psi(\lambda)\right| \\ &\leqslant \sup_{\lambda\in\mathbf{C}} e^{-M(a\frac{m+2}{m+1}|\operatorname{Im}\lambda|)} \left|\lambda^{2n}\Phi(\lambda)\right| \sup_{\lambda\in\mathbf{C}} e^{-M(b\frac{m+2}{m+1}|\operatorname{Im}\lambda|)} \left|\Psi(\lambda)\right|. \end{aligned}$$

Thus the proof is complete. □

Proposition 3.1 allows us to define the #-convolution $T\#\phi$ of $T \in W'_{M,a}$ and $\phi \in W_{M,a}$ as

$$(T\#\phi)(x) = \langle T, \tau_x\phi\rangle, \quad x \in \mathbf{R}.$$

**Proposition 3.3.** *Let $M \in \mathcal{K}$ and $a > 0$. If*

$$T \in W'_{M,a} \quad \text{and} \quad \phi \in W_{M,a},$$

*then $T\#\phi \in C^\infty(\mathbf{R})$ and, for every $k \in \mathbf{N}$, there exists $m \in \mathbf{N}$ such*

$$\left|D^k(T\#\phi)(x)\right| \leqslant Ce^{M(mx)}, \quad x \in [0, \infty).$$

**Proof.** Let $T \in W'_{M,a}$ and $\phi \in W_{M,a}$. According to Proposition 2.6 and by using Hahn–Banach theorem, duality arguments lead to write

$$\langle T, \psi\rangle = \sum_{j=0}^{n} \int_0^\infty e^{M(a\frac{n}{n+1}y)} \Delta^j\psi(y) f_j(y)\, dy, \quad \psi \in W_{M,a},$$

for certain $n \in \mathbf{N}$ and $f_j \in L_\infty(0, \infty)$, $j = 0, 1, \ldots, n$.

In particular, since $\Delta(\tau_x\psi)(y) = \tau_x(\Delta\psi)(y)$, for every $x, y \in (0, \infty)$ and $\psi \in \mathcal{S}_p$, we have

$$(T\#\phi)(x) = \sum_{j=0}^{n} \int_0^\infty e^{M(a\frac{n}{n+1}y)} \tau_x(\Delta^j\phi)(y) f_j(y)\, dy, \quad x \in (0, \infty).$$

It is clear that $T\#\phi$ is an even function.

The mapping defined by

$$\Phi \to \frac{d^k}{dx^k}\big(\varphi_\lambda(x)\Phi(\lambda)\big)$$

is continuous from $W^{M^\times,1/a}$ into itself, for every $x \in (0,\infty)$ and $k \in \mathbf{N}$. Indeed, let $x \in (0,\infty)$ and $k \in \mathbf{N}$. By virtue of [5, Lemma 3.6, (ii)] we have, for a certain $m \in \mathbf{N}$,

$$\left|\frac{d^k}{dx^k}\varphi_\lambda(x)\right| \leqslant C(1+x)^3\big(|\lambda|^2+\rho^2\big)^m e^{(|\operatorname{Im}\lambda|-\rho)x}, \quad \lambda \in \mathbf{C}.$$

Hence, if $s, l \in \mathbf{N}$, then, for every $\lambda \in \mathbf{C}$,

$$\begin{aligned}
&e^{-M^\times(\frac{1}{a}\frac{s+2}{s+1}|\operatorname{Im}\lambda|)}\left|\lambda^{2l}\Phi(\lambda)\frac{d^k}{dx^k}\varphi_\lambda(x)\right| \\
&\quad\leqslant C(1+x)^3 e^{|\operatorname{Im}\lambda|x-M^\times(\frac{1}{a}\frac{s+2}{s+1}|\operatorname{Im}\lambda|)-\rho x}\big(|\lambda|^2+\rho^2\big)^m\left|\lambda^{2l}\Phi(\lambda)\right| \\
&\quad\leqslant C(1+x)^3 e^{-\rho x+M(ax(s^2+3s+2))}e^{-M^\times(\frac{1}{a}\frac{s+3}{s+2}|\operatorname{Im}\lambda|)} \\
&\qquad\times \big(|\lambda|^2+\rho^2\big)^m\left|\lambda^{2l}\Phi(\lambda)\right|.
\end{aligned} \tag{3.1}$$

We conclude that, for every $s, l \in \mathbf{N}$,

$$\begin{aligned}
&q_{s,l}\left(\frac{d^k}{dx^k}\big(\varphi_\lambda(x)\big)\Phi(\lambda)\right) \\
&\quad\leqslant C(1+x)^3 e^{-\rho x+M(ax(s^2+3s+2))}\sum_{j=0}^{m} q_{s+1,l+j}(\Phi).
\end{aligned}$$

Thus we prove that the mapping defined by

$$\Phi \to \frac{d^k}{dx^k}\big(\varphi_\lambda(x)\Phi(\lambda)\big)$$

is continuous from $W^{M^\times,1/a}$ into itself, and, moreover, in agreement with Corollary 2.4, by (3.1) and by taking into account that $M \in \mathcal{K}$, the desired result is established. □

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